MTH241 Fall 2024: Exam 01

Instructor: Dr. Ling Liang

Date: September 25, 2024

Time: 3:00pm - 3:50pm

Name:

UID:

Closed book, no calculator, show your work clearly.

1. (15pt) Mark True or False for each of the following statements. (**Grading:** 3pt each)

   (a) Let P, Q, R be any three points in the space, then $\|\overrightarrow{PQ}\| \leq \|\overrightarrow{PR}\| + \|\overrightarrow{RQ}\|$.
   Your answer: **True** **False**

   (b) Let $\vec{a}$ be a vector and c be any number, then $\|c\vec{a}\| = c\|\vec{a}\|$. $\|-2a\| = 2\|a\|$
   Your answer: **True** **False**

   (c) Let $\vec{a}$ and $\vec{b}$ be nonzero vectors and θ be the angle between them. Then, $\vec{a} \bullet \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin(\theta)$.
   Your answer: **True** **False**

   (d) Any three points in the spaces that are not on the same line uniquely determine a plane.
   Your answer: **True** **False**

   (e) Any smooth curve is also piecewise smooth.
   Your answer: **True** **False**

2. (30pt) Let $\vec{a} = -\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} - 3\vec{k}$.

   (a) Compute $\vec{a} \bullet \vec{b}$, $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. (**Grading:** 4pt for working, 3pt for correct dot product, 4pt for each of the correct cross products)

$a = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

$b = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$a \cdot b = 2 + 1 + 6 = 9$

$a \times b = \begin{vmatrix} i & j & k \\ -1 & 1 & -2 \\ -2 & 1 & -3 \end{vmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$b \times a = -a \times b = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

(b) Compute the projection of $\vec{b}$ onto $\vec{a}$. (**Grading:** 4pt for working, 2pt for correct norm, 4pt for correct projection)

$\text{pr}_a b = \left(\frac{a \cdot b}{\|a\|^2}\right) a = \frac{9}{6} a = \frac{3}{2}(-1, 1, -2)$

$a = \langle -1, 1, -2 \rangle \rightsquigarrow \|a\| = 6$

(c) Find an equation of the plane that contains a point $P_0 = (-1, -2, -1)$ and is parallel to both $\vec{a}$ and $\vec{b}$. (**Grading:** 5pt for working, 5pt for correct equation)

$N = a \times b = (-1, 1, 1)$

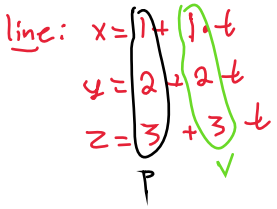
$P_0 = (-1, -2, -1)$

$\Rightarrow E: -(x+1) + (y+2) + (z+1) = 0$

3. (30pt) Consider the following equations for the line ℓ :

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}.$$

(a) Find a point on the line and a vector that is parallel to the line. (**Grading:** 3pt for working, 6pt for correct point, 6pt for correct vector)



$P=(1,2,3)$ pt in line

$v=(1,2,3)$ vector // line

(b) Let $P_1=(1,1,1)$, find the distance D from P_1 to the line ℓ. (**Grading:** 5pt for working, 5pt for correct formula, 5pt for correct distance)

$$d(P_1,\ell)=\frac{\|P_1P\times v\|}{\|v\|}=\frac{\|(0,1,2)\times(1,2,3)\|}{\|(1,2,3)\|}=\frac{\|(-1,2,-1)\|}{\|(1,2,3)\|}=\frac{\sqrt{6}}{\sqrt{14}}=\sqrt{\frac{3}{7}}$$

$P_1=(1,1,1)$
$P=(1,2,3)$ $\rightsquigarrow P_1P=(0,1,2)$
$v=(1,2,3)$

$$P_1P\times v=\begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}=\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

4. (25pt) Let $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + \frac{2}{3}t^{3/2}\vec{k}$ for $t \in [0,1]$ be the parameterization of a curve C.

(a) Determine if C is smooth on $[0,1]$. (**Grading:** 4pt for working, 5pt for correct criterion of smoothness, 3pt for correct answer)

$r'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ \sqrt{t} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ b/c $\sqrt{t} > 0$ for all $t \in (0,1)$

$r'(t) \neq 0$ for any $t \in (0,1)$

- $r'(t)$ is continuous on $[0,1]$

Hence $r(t)$ is smooth

(b) Compute the length L of the curve C defined on $[0,1]$. (**Grading:** 5pt for working, 5pt for correct formula, 3pt for correct distance.)

$$L = \int_0^1 \|r'(t)\|\, dt = \int_0^1 \sqrt{\cos^2 t + \sin^2 t + \sqrt{t}^2} = \int_0^1 \sqrt{1+1+t} = \int_0^1 \sqrt{1+t} = \frac{2}{3}(t+1)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right)$$

Extra page: